Transforming the Way We Teach Function Transformations

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Using transparencies helps students learn function transformations—through understanding, not memorization.

Recently, when one of our mathematics classes began studying transformations of functions, we inadvertently fell back into an old habit. In implementing the ideas of NCTM's *Principles and Standards for School Mathematics* (2000) in our classrooms, we have tried to move from rote manipulations of symbols to a more conceptual approach. That is, instead of having students memorize formulas and procedures, we focus on the underlying concepts so that students can know *why* procedures work, not simply *how* they work. But now, as we began teaching function transformations, we found ourselves simply repeating the rules we had learned as students.

Our students, having learned from previous lessons always to ask "Why?" recognized that we were covering transformations of functions differently from previous topics: We were simply giving them procedures to apply. Dissatisfied, they asked, as many students do, "Why do we do the *opposite* when a change is made inside the parentheses. Why does $f(x - 2)$ move the graph of f two units to the right?"

Of course, we can simply plot the graphs of several transformations of functions using tables of values. However, previous experience shows that even though the appropriate patterns can be made to appear in the table, the effect of this "motion" is not as direct as in a graphical demonstration. So, to answer this question in a more pedagogically satisfying manner, we went back to basics: What is the basic idea of a function, and how do the various transformations act on this meaning?

INPUTS AND OUTPUTS

A *function* is a well-defined rule that relates inputs to outputs. The common conceptual metaphor for a function is a machine: We put input values (x) into the machine, we apply a rule, and the machine produces output values (y). This conception, of course, leads to the common definition of the *graph* of a function—the set of all input and output pairs, written as coordinates (x, y).

This definition, although standard, is also essential for a conceptual understanding of function transformations. When realizing a function in this manner—as a rule that relates the input to the output—we can classify function transformations as affecting either the input or the output. Transformations such as $f(x) + 3$ and $2f(x)$ have arithmetic operations being performed on the values of $f(x)$—that is, the output values. For example, $f(x) + 3$ adds 3 to each output value, or y-value, but leaves the x-values unchanged. This transformation affects the graph of $f(x)$ by shifting it upward as each y-value is moved up 3 units (see **fig. 1**). Likewise, $2f(x)$ doubles the output values (the y-values) but, once again, leaves the input values (the x-values) unchanged. Geometrically, this transformation can be seen on the graph as a vertical stretch (see **fig. 2**).

Transformations to the output are relatively easy for students to grasp because the effects on the graph are consistent with their intuitive expectations. For example, adding 3 to the outputs coincides graphically with the standard notion of "positive means up." Similarly, subtracting 3 from the outputs would easily translate to moving the graph down 3 units. However, transformations to the input tend to give students more difficulty conceptually, most likely because the effects of transformations on the input seem to be counterintuitive. Teachers have often tried to explain away this uneasiness by telling students to "do the opposite operation when the change is inside the parentheses." This approach, of course, leaves much to be desired, especially when the goal of effective mathematics instruction is to help students develop reasoning about *why* things happen.

A PEDAGOGICAL METHOD FOR TEACHING TRANSFORMATIONS

Function transformations are completely described by their roles on either outputs or inputs. Accordingly, we like to view the graphs of functions using two components: a blank grid printed on a piece of paper (see **fig. 3**) and a function graph drawn on a transparency without gridlines (see **fig. 4**). These components can be seen in **figure 5**. The graph on the transparency represents the output values of the function, whereas the x-axis on the blank grid represents the input values. Hence, the motion of a func-

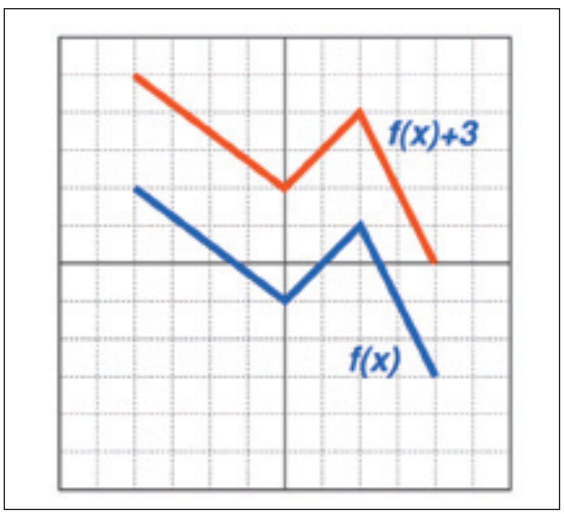


Fig. 1 The red graph is obtained by shifting the blue graph up 3 units.

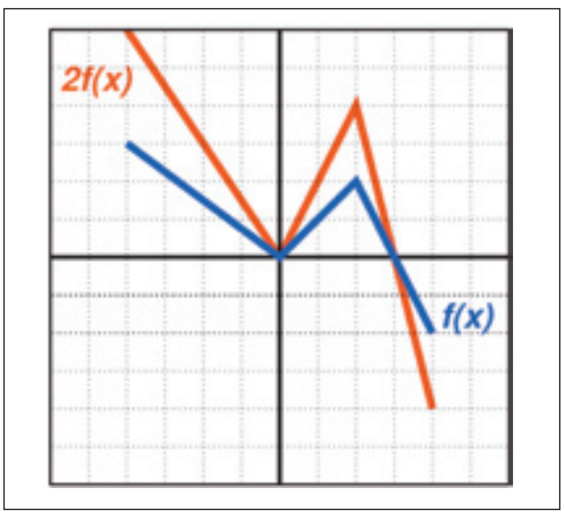


Fig. 2 The red graph is obtained by stretching the blue graph vertically by a factor of 2.

Function Input Grid

Directions: Copy the grid onto paper and place under the transparency from Function Output Axes.

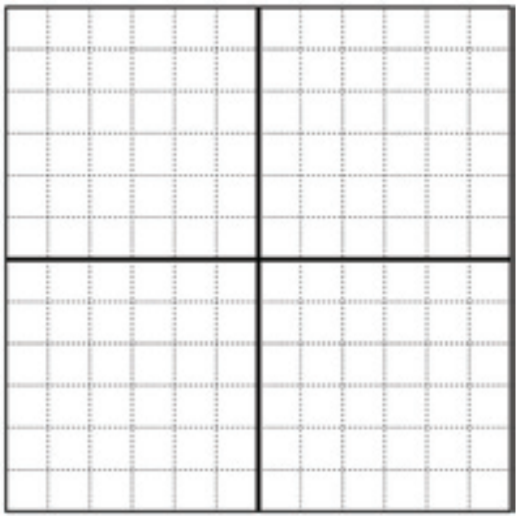


Fig. 3 In this approach to understanding function transformations, students' first component is a grid.

Function Output Axes

Directions: Copy the axes onto a transparency and draw desired graph directly on the transparency.

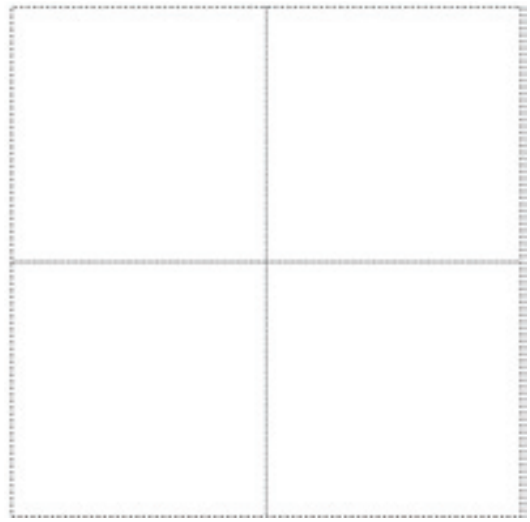


Fig. 4 The gridless transparency serves as a receptacle for function graphs.

tion transformation can be modeled by moving the appropriate visual aid. If the function transformation is an output transformation, we move the graph; if it is an input transformation, we move the grid.

Consider, for example, the transformation $f(x) + 3$. First, we ask students to identify the transformation as input or output. Once they correctly identify it as an output transformation, we discuss which of the materials will be moved and how. The outputs are being increased by 3, so we must move the graph up 3 units, as in **figure 6**. Other examples of this type of transformation are of the form $f(x) + k$ or $f(x) - k$, where addition corresponds to moving the graph up and subtraction corresponds to moving the graph down. In both cases, the key point is that the motion is on the transparency graph, not the paper input grid.

As another example, consider the transformation $f(x - 2)$. Again, we begin the lesson by asking students to identify the type of transformation that is occurring. Once they correctly identify it as an input transformation, we can discuss how it affects only the paper grid underneath the transparency graph. The transformation $f(x - 2)$ is a transformation in which the input is being decreased by 2, meaning that every value of x is decreased by 2 before being input into the function. This transformation can be represented visually by moving the

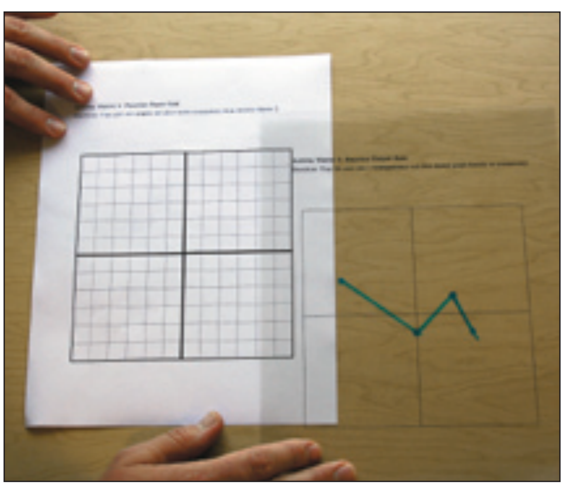


Fig. 5 The transparency of the function output axes can be placed on the grid as needed.

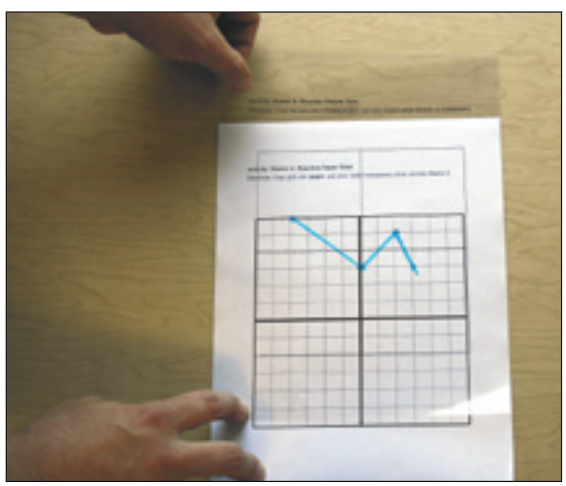


Fig. 6 Only the transparency containing the graph has been moved.

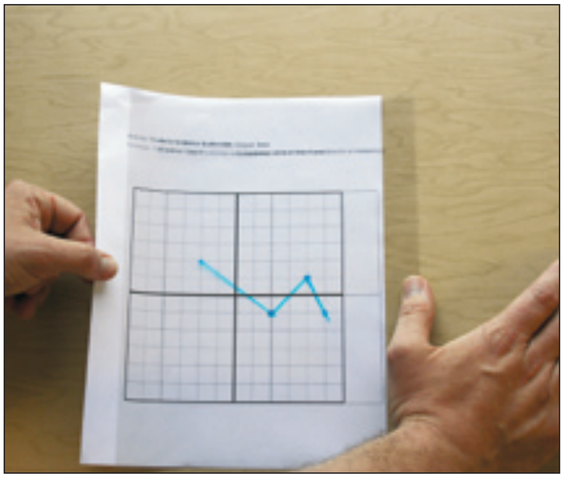


Fig. 7 Because a horizontal shift is a transformation only on the input values, we leave the graph of *f* alone as we move the grid.

Table 1 Horizontal Shifts

x	-4	-3	-2	-1	0	1	2	3
$f(x)$	2	1.25	0.5	-0.75	-1	0	1	-1
$f(x-2)$			2	1.25	0.5	-0.75	-1	0

blank grid 2 units to the left underneath the graph of *f* (see **fig. 7**). This act of moving the input values 2 units to the left underneath the graph of *f* makes the graph of $f(x)$ appear to shift right. Similarly, transformations of the form $f(x + h)$, $h > 0$, would make the graph appear to shift to the left because the grid underneath would be moved to the right. In both cases, students can see the desired effect without having to suspend their intuition about signed numbers.

Further, we can take this opportunity to explore what happens to $f(x - 2)$ numerically by creating a table of input and output values for $f(x)$. By also creating an entry for the values of $f(x - 2)$, we can see the apparent shift of the values of $f(x)$ 2 units to the right (see **table 1**). This investigation, in alignment with NCTM's *Principles and Standards for School Mathematics* (2000), provides both graphical and numerical evidence for the effect of an additive input transformation.

IMPLEMENTATION

This pedagogical method works well with additive transformations—that is, transformations that involve only addition and subtraction. For multiplicative transformations, we need the ability to stretch and compress axes, and our simple transparency technique fails. In our classrooms, we use the transparency technique to cover additive transformations and generate discussion about multiplicative transformations. The same basic ideas apply. The transformation affects either the input or the output, but in this case we have to use a bit more visual imagery to express the stretching or compressing effects of the multiplication. In this discussion we tend to focus on rubber bands, balloons, and so forth.

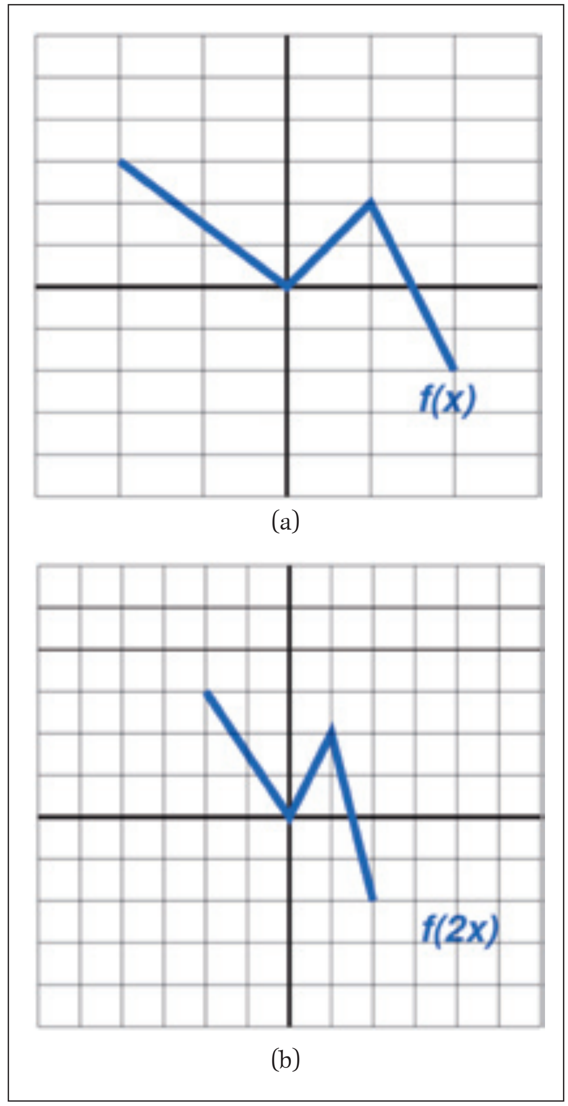


Fig. 8 With $f(2x)$, we first stretch the grid (**a**) and then relocate the points (**b**).

However, concrete visual aids can be used in this context. As an example, consider the multiplicative transformation $f(2x)$. Through discussion, it should become clear that this transformation involves stretching the input grid horizontally by a factor of 2 and then placing the graph of $f(x)$ onto this grid (see **fig. 8a**). Points on the graph that were at $x = 2$ are now located at $x = 1$ with respect to the new, stretched grid. If we compute the new location of each point's x-coordinate in this manner and plot each on the original, unstretched grid, the transformed graph will appear to have been compressed horizontally by a factor of 2 (see **fig. 8b**). Once again, the effect of input transformations on the graph is opposite that predicted by the operation because the transformation is applied to the underlying input grid, not to the graph itself.

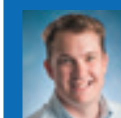
This discussion of function can be adapted for several levels of mathematics. We have successfully used the technique in high school algebra, college algebra, trigonometry, calculus, and even graduate-level statistics. Student interest has been overwhelming, with most students identifying the visual aspects of the technique as being their favorite. The technique too is flexible, allowing teachers to modify the technique to satisfy the needs of all students.

CONCLUSION

We have found that by using the input-output definition of *function*, we can examine transformations of functions simply by looking at changes to input or output and the respective changes to the graph. Applying transformations to the input and output values and examining the resulting graphs helps students build a better understanding of functions. Most important, this technique allows students to keep their intuitive notions of number and operations without having to memorize counterintuitive procedures.

REFERENCE

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

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