Teaching Integer Arithmetic without Rules: An Embodied Approach

By Thomas J. Faulkenberry and Eileen D. Faulkenberry

When teaching mathematics courses for prospective mathematics teachers, we spend a great deal of time trying to help these future teachers develop a broad conceptual understanding about the most basic concepts of arithmetic. In accord with the expectations of NCTM's *Principles and Standards for School Mathematics* (2000), we use questions like "What does the addition sign really *mean*?" and "How many different ways can you assign meaning to the concept of division?" to get our preservice teachers to think more deeply about the mathematics they will be teaching. The class discussion generated from these questions begins to acquaint the future teachers with the ideas of Cognitively Guided Instruction (Carpenter et al., 1999).

As an example, consider the problem 7 - 3. We asked a class of pre-service mathematics teachers to model this arithmetic problem with a word problem. Out of 32 students, all constructed a problem that involved starting with seven objects and removing three of them to see how many were remaining. Typical examples included, "There are seven birds in a yard and three fly away. How many are left?" In the language of Carpenter et al. (1999), this model is an example of the problem type *Separate (result unknown).* We tried to generate some more examples in class, but everyone agreed that any model for 7 - 3 would look similar to ones they had already done.

Of course, there are other models for 7 - 3. Consider the following example:

"Jim has 3 golf balls. Sidney gives him some more golf balls. Now Jim has 7 golf balls. How many did Sidney give to Jim?"

Carpenter et al. (1999) identified this problem type as *Join (change unknown).* It is a perfectly good model for 7 - 3. However, the question remains: *Why did none of our students think of examples that would be classified as a join problem?*

The same issues regarding the formation of meaning in mathematics also come up when we work with in-service teachers. During a recent dialogue with a group of in-service teachers at a professional development workshop, the topic of integer arithmetic came up. Specifically, one of the teachers asked, "How do we teach integer arithmetic so that it *makes sense* to the students without them having to memorize a bunch of rules?" We proposed using the *Balloons and Sandbags* activity (Herzog, 2008). This activity is easy to understand, requires few classroom materials, and it is based on a principle from cognitive psychology known as *embodied mathematics.*

Embodied Mathematics

Embodied mathematics is a natural outgrowth of scientific work in embodied cognition (Pecher & Zwaan, 2005; Wilson, 2002). It was brought to a wider audience with the work of G. Lakoff and R. Nunez (2000) in their book, *Where Mathematics Comes From.* Lakoff and Nunez make a strong case that all mathematical thought, even the most advanced mathematics, is a result of humans using *conceptual metaphors* to conceptualize abstract thought in terms of embodied, sensory-motor experiences. In other words, we think about abstract mathematical concepts in terms of much simpler, body-based experiences. To understand how this works, let's first consider a non-mathematical example of a conceptual metaphor.

A conceptual metaphor (Lakoff & Johnson, 1980) is not exactly the same thing as a figure of speech (a literary metaphor). Rather, a conceptual metaphor is a way of thinking about something abstract in terms of something else with which we have prior sensory-motor experience. For example, consider the metaphor AFFECTION = WARMTH. As the reader may recognize, this metaphor is at the root of many conventional American English expressions, such as:

- "She *warmed* up to him"
- "We haven't *broken the ice* yet"

Notice that in these linguistic examples, there is a strong correspondence between the structure of the abstract idea *affection* and the embodied idea of *warmth*. Affection is conceptualized as warmth, whereas disaffection is conceptualized as the opposite of warmth, or cold.

How do these ideas apply to mathematical thought? Lakoff and Nunez (2000) present many conceptual metaphors that they believe form the basis for our mathematical ideas. For the purposes of this article, we will only focus on one metaphor that underlies simple arithmetic: *arithmetic = object collection*. With this metaphor, the ideas of simple arithmetic (order, addition, subtraction) are thought of in terms of the embodied experience of collecting objects. Specifically, the structured metaphorical correspondence is listed in Table 1.

Object Collection	**Arithmetic**
Collections of objects the same size	Numbers
The size of the collection	The size of the number
Bigger	Greater
Smaller	Less
The smallest collection	The unit (One)
Putting collections together	Addition
Taking a smaller collection from a larger collection	Subtraction

Table 1: The conceptual metaphor "ARITHMETIC = OBJECT COLLECTION"

Notice that this seems to be precisely what is guiding the *Join/Separate* principles in Carpenter et al. (1999). In the language of embodied mathematics, addition is simply a metaphorical extension of the embodied experience of putting two collections together (joining), and subtraction is a metaphorical extension of the embodied experience of taking a smaller collection away from a larger collection (separating).

At this point, we should have some insight to our original problem: why did none of our students think of the "addition with a missing addend" model for 7 - 3? Using the ideas of embodied mathematics, we posit the following explanation. Because the students are seeing the operation as subtraction, and since subtraction is conceptualized (via the ARITHMETIC = OBJECT COLLECTION metaphor) as a process of *taking away*, each of the students' models involved taking away. There is no reason, cognitively, for a subtraction problem such as 7 - 3 to have been modeled as $3 + N = 7$, since the latter is perceptually an *addition* problem. Mathematically, they are the same, but *cognitively*, they are completely different.

Based on this basic arithmetic metaphor, does it follow that integer arithmetic is structured cognitively as object collection? Integer arithmetic is often difficult for students of all ages to master because it seems to be just a collection of rules; that is, it lacks an embodied foundation. For example, to calculate 3 - (-5), one simply changes the two negatives to a positive and adds 3 and 5 to get 8. This, of course, is one of the easier rules; the rest can be even more difficult or confusing to students. So, when teachers approach us about how to teach integer arithmetic for understanding using embodied mathematics, we often look to the *Balloons and Sandbags* activity (Herzog, 2008).

Embodied Integer Arithmetic with Balloons and Sandbags

Balloons and Sandbags is a conceptual model of integer arithmetic where each balloon represents a positive quantity and each sandbag represents a negative quantity. The idea stems from the physical interpretation that a balloon lifts upward and a sandbag would fall to the ground. The basic idea is that each balloon exerts an equal and opposite force to each sandbag. So, for example, an addition problem such as $3 + (-2)$ could be modeled as starting with three balloons (an upward lift of 3 units) and adding two sandbags (a downward pull of 2 units), resulting in a net upward lift of 1 unit (see Figure 1). Hence, $3 + (-2) = 1$. Notice that with this representation, the operation of addition is still conceptualized as "putting two collections together." Contrast this with the "rule" which states "a plus and a minus together equals a minus, so it's three minus two." In this case, a student is forced to give up his or her embodied conception of addition as "joining" in favor of a rule with no reason. At best, this perplexes the student; at worst, it reinforces the negative stereotype that mathematics is just a bunch of rules.

Any integer addition or subtraction problem can be modeled with the Balloons and Sandbags activity. To guide our discussion, we consider two main classes of problems: problems involving addition (joining) and problems involving subtraction (separating).

Addition Problems

For example, consider the problem $-4 + 7$. This problem would be modeled by starting with 4 sandbags (to represent downward pull of 4 units embodied in the integer -4). The action that occurs is adding 7 balloons. Since every balloon exerts an equal and opposite force on every sandbag, the 4 sandbags pair with 4 of the newly added balloons so that each pair gives a net lift of 0 units (see Figure 2). All that remains are 3 balloons, which each have positive lift. Hence, the answer is $-4 + 7 = 3$.

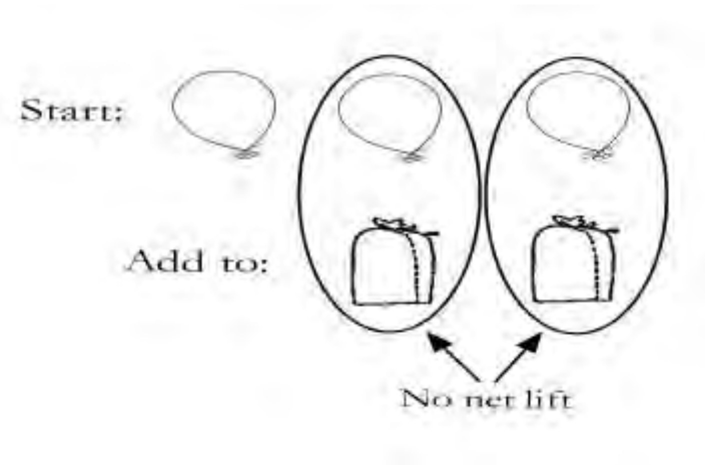

Figure 1. Balloons and sandbags representation of 3 + (-2).

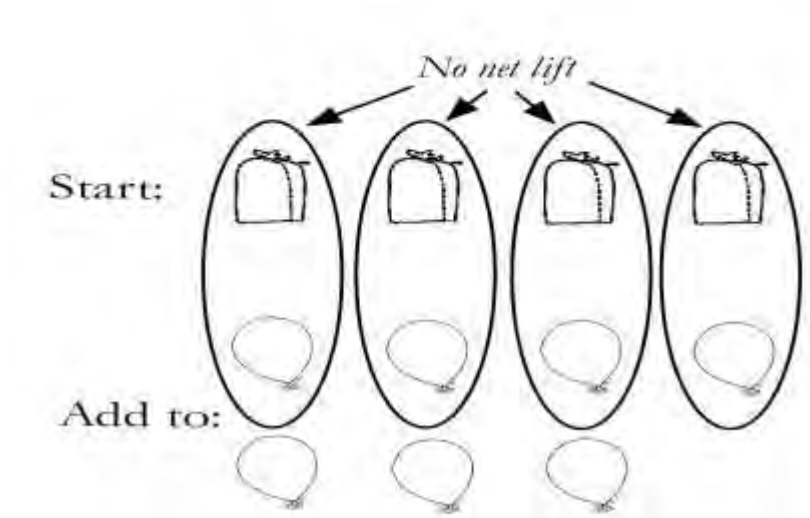

Figure 2. Balloons and sandbags representation of -4 + 7

As another example, consider –3 + (-4). Again, this problem would start with 3 sandbags, to which we would add 4 more sandbags (see Figure 3). This gives a total of 7 sandbags, resulting in a net lift of –7. Hence, the answer is –3 + (-4) = -7.

Subtraction Problems

For example, consider the problem 3 - (-5). Note that even though the popular strategy for solving this problem involves changing this to 3 + 5, this is technically a subtraction problem. Using the Balloons and Sandbags model, we may model this by starting with three balloons. The problem is asking us to remove five sandbags from this collection. This is impossible to do since we are only starting with three balloons. This requires the addition of what we technically term "zero pairs;" that is, balloon/sandbag pairs with a net lift of 0 units. Through class discussion, we discuss what different configurations of balloons would result in the same net lift as three balloons would; for example, starting with 3 balloons, no number of zero pairs that we add to this collection will change the net lift of 3 units provided by these 3 balloons. Specifically, 3 balloons with 5 balloon/sandbag pairs would still exert a net lift of 3 balloons. However, this configuration does indeed have 5 sandbags to remove; if we remove them, we are left with 8 balloons (see Figure 4). Hence, 3 - (-5) = 8. Once again, the "taking away" conception of subtraction is preserved, as opposed to the rule that says that this subtraction problem is really an addition problem. Note, however, how the solution of this problem through the Balloons and Sandbags activity does indeed give some insight as to why this rule works in general, giving the student a chance to discover the rule on his/her own. Since the student added five balloon/sandbag pairs to the configuration in order to have five sandbags to remove, the idea of addition has been introduced to the situation. Also, since the sandbags were removed but the balloons remained, the student actually added the opposite (the balloons) of what he wanted to take away (the sandbags), thus leading to the idea of subtraction being equivalent to "adding the opposite."

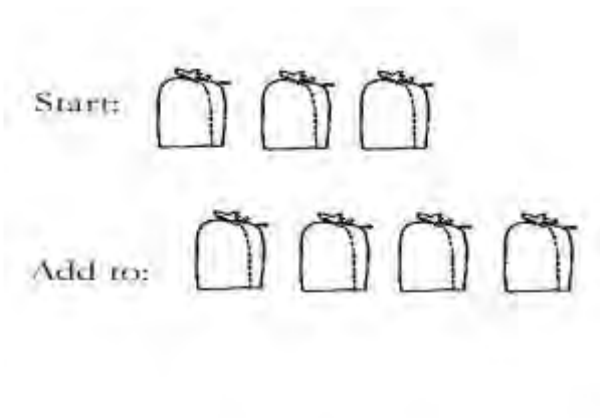

Figure 3. Balloons and sandbags representation of -3 + (-4)

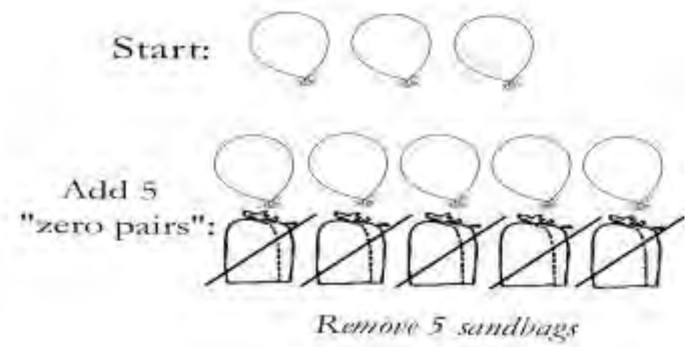

Figure 4. Balloons and sandbags representation of 3 - (-5)

As another example illustrating the technique of adding zero pairs, consider the example –3 – 4. The more mathematically sophisticated student will notice that this problem has the same answer as the problem –3 + (-4) that we solved above. However, let us model the problem as it is presented: as a subtraction problem. The problem states that we start with 3 sandbags, and the intended action is to remove 4 balloons. Since we have no balloons, we need to introduce them without changing the net lift of the system; again, we can accomplish this by adding zero pairs. To be able to remove 4 balloons, it is sufficient to add 4 balloon/sandbag pairs. After we remove the 4 balloons, one can see that we are left with 7 sandbags (see Figure 5). Hence, the solution to the problem is –3 –4 = -7.

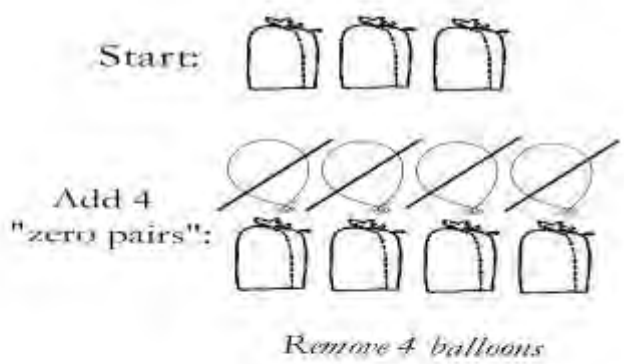

Figure 5. Balloons and sandbags representation of -3 - 4

Classroom Issues

The Balloons and Sandbags model is easy to use in the classroom. The only materials needed are something to represent balloons and something to represent sandbags. For children just beginning to investigate integer arithmetic, teachers may decide to provide pictures of balloons and pictures of sandbags that are copied onto cardstock and cut out with scissors. Children may then model their problems through manipulating these concrete materials in the manner discussed above. Children who have had some exposure to this activity with concrete materials will begin to solve problems by simply drawing their own balloons and sandbags on paper and recording their actions (see Figures 6 and 7). In Figure 6, a sixth-grade student is solving the problem -2 + 6. She begins with two sandbags to represent the -2. Then she adds 6 balloons. Next, she circles pairs of balloons and sandbags to create "zero pairs." Once she has made all the zero pairs she can, she knows that the remaining pieces represent the answer. She has four balloons left so writes her answer as 4.

Figure 7 shows an eighth-grade student solving a subtraction problem, 3-8. She begins with three balloons. She writes that she needs to take away 8 balloons so she must add balloon/sandbag pairs. Observing the student, we asked her how she knew how many balloon/sandbag pairs to add. She said she knew she needed 5 pairs because 3 + 5 = 8. Once the student had 8 balloons to remove, she crossed the balloons out leaving five sandbags.

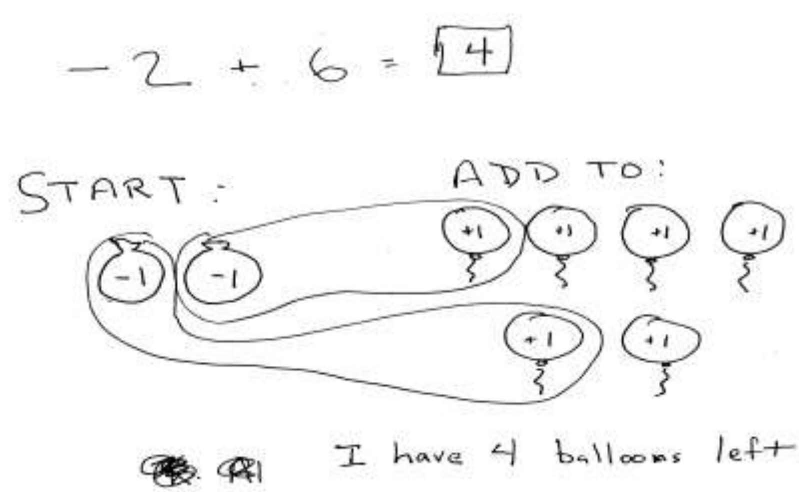

Figure 6. An example of a 6th grader's work on -2 + 6

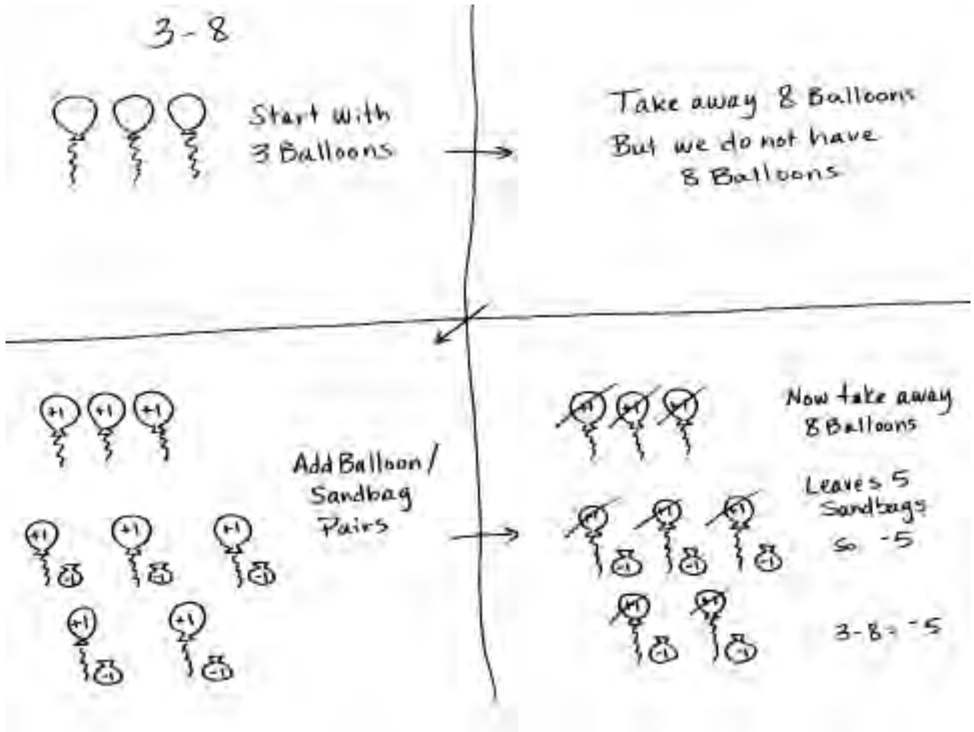

Figure 7. An example of an 8th grader's work on 3 - 8

Conclusion

These methods of teaching integer arithmetic are only a small sample, but they are indicative of a class of methods that appeals not only to good pedagogical sense, but also to current research in cognitive psychology. Some people may argue that the Balloons and Sandbags model is simply another set of rules for adding and subtracting integers. While there are certainly some "rules" for implementing the Balloons and Sandbags model correctly, these rules are based on the embodied notions of addition as "joining together" and subtraction as "taking away from." Contrast this with some of the other popular sets of rules for integers that involve switching signs according to certain patterns. These rules may often be efficient, but this efficiency comes at the expense of contradicting a student's embodied sense of arithmetic.

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