Constructivism in Mathematics Education: A Historical and Personal Perspective

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INTRODUCTION

In this paper, we discuss the historical roots and the practical uses of constructivism in the mathematics classroom. Constructivism is the popular, yet mildly controversial belief that students construct their own knowledge through self-modification of cognitive structures. This self-modification is a largely unconscious, yet goal-directed, process by which the student reacts to a cognitive disturbance by changing how he or she thinks about a concept to accommodate the novel piece of information, thus relieving the cognitive disturbance. Essentially, this means that when the student encounters a hard problem, the student (ideally) reacts by thinking about it until it makes sense. This challenges the classic behaviorist model where a student is presented with stimuli (problems, exercises, etc.) and shown how to achieve a certain response. The behaviorist model requires some sort of external reward. In contrast, the main tenet of constructivism is that no external reward is necessary; rather, the "comfort" of the newly modified cognitive structure is rewarding in itself.

Constructivism is a part of several psychological theories. The historical roots of constructivism as a psychological theory are most commonly traced to the work of Jean Piaget, although there are some elements of Piaget's constructivism that come from the early Gestalt psychologists.

As collegiate mathematics education teachers and researchers, we have much experience with constructivism as both a research paradigm and a teaching method. The success of constructivism, both as a pedagogical technique and as a psychological theory, provides converging evidence of its utility. We now discuss these two facets in detail.

CONSTRUCTIVISM AS PEDAGOGY

The view of constructivism as a psychological theory tells us much about how students learn mathematics. Using this information, many teachers have begun to think about exactly how they conduct their mathematics classrooms. The standard model for mathematics teaching has long been the lecture, as exemplified in Krantz (1999, p. 12), where he says, "Lectures have been used to good effect for more than 3000 years." While no one will probably deny that they have seen some very effective lectures in their educational experience, the modern thought is that a good majority of lectures tend to be rather ineffective, especially in the mathematics classroom. As Dubinsky (1999) points out, how do we really know what the classroom style of Newton was like? There seems to be no historical documentation pointing to the exact teaching style of these great mathematicians.

The use of constructivism in the mathematics classroom has many variations. The one thing that these variations have in common, however, is the central role of the *student* in the learning process. In the following, we will present some examples of classroom events that demonstrate the student-centered constructivist approach.

Example 1 – Alphabitia

When we teach mathematics courses for elementary teachers, one of the topics that our students encounter in alternative bases for numeration. A classic activity for investigating this topic is Tom Bassarear's Alphabitia activity (Bassarear, 2005). In this activity, the students play the role of archaeologists who "dig up" the ancient civilization of Alphabitia. From some artifacts they determine that the Alphabitians used a numeration system that consisted of only the symbols A, B, C, D, and 0. The task for the student archaeologists is to figure out exactly how the Alphabitians were able to represent numbers with only these symbols. Of course, those readers with experience in base-5 arithmetic will immediately see a way to do this, but our students almost always are initially baffled. We spend about a week trying different systems until we eventually converge on the canonical base-5 positional place value system. It is a long road for the students, but their satisfaction with their ability to construct something "brand new" is priceless.

Equilibration

This activity is good on many fronts, but it is especially good since it quickly confronts the student with a novel situation that they have to mentally organize. This cognitive organization process is called *equilibration*. Equilibration is the process by which a learner attempts to organize a new piece of information by placing it into his or her current cognitive structure and modifying that cognitive structure accordingly. Equilibration is closely related to the Gestalt concept of *harmonious equilibrium*, where consciousness tends to move away from "uncomfortable" stimuli toward a more "comfortable" state. Equilibration is central to Piaget's constructivism, as evidenced by his claim that it is the organizing principle of cognitive development (Dubinsky & Lewin, 1986). Piaget's notion of equilibration is a cyclic process. If an encountered piece of knowledge is novel in the sense that it doesn't fit with the learner's current cognitive framework, the learner's cognitive system is now out of equilibrium. This is called *disequilibration*.

In the Alphabitia activity above, the students immediately undergo disequilibration when presented with the problem of reinventing a numeration system. The purpose of spending a week on the activity is to allow time for the students to modify their existing cognitive structures to *assimilate* or *accommodate* this new material.

Example 2: Alphabitia continued

Once the initial activity of Alphabitia is completed, the students are satisfied with their ability to construct new representations for numbers. However, their knowledge is still quite limited at this point, as evidenced when the extension to the activity is given. In the extension, we ask the students to then find a way to add and subtract these new Alphabitian numerals. Once again, the students are faced with a novel situation that they don't yet know how to handle. So, more time is spent with the students working in groups until they find a way to accommodate this new problem.

Reflective Abstraction

Notice that the students once again need to accommodate a new piece of information. This time, however, their cognitive structures are working on a higher level than before. This cognitive reconstruction is called *reflective abstraction*, as it involves reflecting the existing cognitive structures to a higher plane of thought and applying these structures to new stimuli. This is sometimes called "generalization" or "extension." A more extreme version of reflective abstraction can be found in the next example.

Example 3: Alphabitia concluded

After two weeks of work on the Alphabitian numeration system, our students have gone from an unfamiliar numeration system to adding and subtracting in this new numeration system. Of course, the students have actually just computed a group structure. As such, the numeration system is no longer just a set of isolated processes; rather, it has become a complete *system*. Once the students are able to view Alphabitia as a system, they have undergone the most radical cognitive reconstruction: *encapsulation*.

Encapsulation

Encapsulation is the most interesting (and extreme) form of reflective abstraction (for mathematics education). In encapsulation, implicit processes are coalesced into a whole unit, on which more actions and processes can be performed. In other words, *encapsulation is the conversion of a dynamic process to a static object* (Dubinsky, 1991).

Example: An APOS Analysis of the Concept of Function

APOS Theory (Asiala, et al., 1996) is a psychological and educational theory of how students learn mathematics. The acronym APOS represents *Action, Process, Object, & Schema;* a cycle of conceptual levels that a mathematics student progresses through when building a set of (cognitive) organizing principles (a *schema)* about a particular topic. We illustrate the APOS cycle by considering how students typically learn the concept of *function*.

When a student first learns about functions, the student is undoubtedly reminded that a function is a "machine" or formula that transforms a number that one "plugs in" to give a new number. In other words, the function is something the student performs *actions* on to get an answer. After the student does a few problems and begins to reflect on these actions, the student should begin to see the function as a complete *process* of actions; i.e., first, plug in the number, then simplify the algebraic expression to get the result. This cognitive transformation is called *interiorization*. See Figure 1 for an illustration.

It is worth noting at this point that the *process* conception of function is the conceptual level that most college algebra students attain after a one-semester course. Once these students move on to calculus, however, they need to be able to take a derivative of a function. A derivative is nothing more than a higher-level function that takes functions as inputs. Since, at a conceptual level, a functional input needs to be an *object*, the concept of function that the calculus student possesses must go beyond the *process* conception of function; that is, the dynamic *process* of function must be encapsulated to become a static *object*. When the student realizes that a function is an object, just like a number, the student can then try to extend number relationships to function relationships, which include being able to use a function as an input to another function like the derivative.

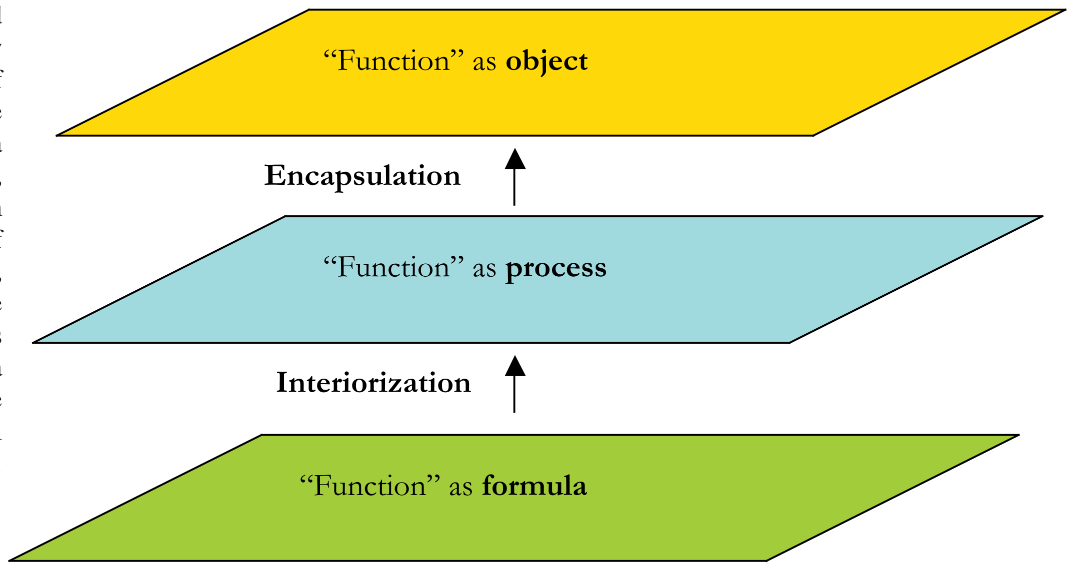

Figure 1. Illustration of *interiorization*.

Example: Calculus

Another example of the APOS cycle is found in studying the Fundamental Theorem of Calculus. For a beginning calculus student, an integral is often viewed as a process for computing the area underneath a curve; for example,

$$\int_1^2 \frac{1}{x}\,dx$$

is equal to the area underneath the curve $f(x) = \frac{1}{x}$ from

$x=1$ to $x=2$. This conception is at the action and process level. A few weeks later, however, the student encounters the Fundamental Theorem of Calculus, of which a certain form deals with functions defined as integrals, such as the familiar

$$\ln(x) = \int_1^x \frac{1}{t}\, dt .$$

Students often have trouble with a representation of this type. This is most likely because an integral such as the former one is computed using a process, whereas in the latter, the integral is part of the function definition. What seems to be lacking is the encapsulation of this area process to an object that can have its parameters vary (Dubinsky, 1991).

Recent research in mathematics education has indicated that successful mathematics performance occurs whenever students are able to encapsulate dynamic processes into static objects. The ability to encapsulate processes into objects is generally considered to be very difficult (Sfard, 1991). As an example, consider how students think about the equality 0.99999....=1. Many students believe this to actually not be true, although it is a well-known result that can be found in many mathematics textbooks. The difficulty seems to be stemming from the students' inability to encapsulate the dynamic process of "repeating the 9's indefinitely" to a static object – the limit of the infinite process (Weller, et al., 2004).

CONSTRUCTIVISM AS A PARADIGM FOR RESEARCH IN MATHEMATICAL UNDERSTANDING

"Understanding" is a concept that is part of everyone's folkloric knowledge about teaching, but it is also a term that tends to lack an operational definition. Up until 1978, a student's mathematical understanding was equated with the student's (algorithmic) knowledge of mathematics (Meel, 2003). After 1978, understanding came to be realized as more organic than algorithmic knowledge, and several categories of understanding were proposed. Richard Skemp (1976) was one of the first researchers of mathematics education who started to use knowledge from cognitive psychology to inform what was going on in the mathematics classroom. While his theory of understanding was not purely constructivist (in the sense of Piaget), it did certainly contain elements of constructivism. For example, Skemp's levels of understanding each contained a reflective subcategory, which was akin to Piaget's reflective abstraction.

Following Skemp's lead, many researchers in mathematics education have proposed theories of mathematical understanding that explicitly use ideas from constructivism. Examples include the concept image and concept definition model of Tall and Vinner (1981), the multiple representations model of Kaput (1989), and the growth of understanding model of Pirie and Kieren (1994). See Meel (2003) for a more complete description of these models. Constructivist models are not limited to undergraduate mathematics, however, as these models may inadvertently suggest. Kamii (2000) has done extensive work with young children and their conceptions of arithmetic using a constructivist model.

PERSONAL PERSPECTIVES ON CONSTRUCTIVISM

We conclude by outlining two approaches to constructivism in the mathematics classroom: one is a formal approach developed by Ed Dubinsky, and the other is our personal amalgamation of these techniques.

THE ACE TEACHING CYCLE

The ACE Teaching cycle was developed by Ed Dubinsky and is outlined very nicely in Asiala, et al. (1996). The acronym ACE represents the three components of the cycle: activities, class discussion, and exercises. In the activity portion of the cycle, the students work in groups (often in the computer lab if appropriate) on tasks that are specially designed to help the students develop the correct cognitive constructions suggested by the constructivist model; that is, to help the students encapsulate processes into objects. These activities may last for more than one class period. At the end of the activity, the groups come together for a class discussion period, where the instructor leads discussion among the groups. The purpose of this discussion is to provide a medium for the students to begin the process of reflective abstraction. The role of the instructor is to help the students successfully tie things together. Finally, out-of-class exercises are assigned for the students to work on in teams. The exercises are used to help the students reinforce their conceptual framework of the mathematics being studied.

This style of teaching does not lend itself well to a standard textbook, however. One extreme solution has been proposed by Asiala, et al. (1996), whose solution is to write new textbooks that support this teaching cycle. Realistically, this is not possible for most instructors, so each instructor must find his or her own technique for using a textbook to support this form of instruction. Some find it necessary to abandon the textbook completely, where others supplement the textbook with other materials. As an example, we will now present our method of constructivist teaching that has worked well for both of us.

OUR PERSONAL STYLE OF CONSTRUCTIVIST TEACHING

In our mathematics classrooms, the students are frequently presented with a problem situation to solve. This involves finding rich mathematical tasks such as the Alphabitia activity. The instructor gives a brief introduction to the problem; then the students work in small groups. The introduction is not to provide direction in the solving of the problem but to make sure all the students understand the nature of the problem. The instructor then observes the small groups, often asking questions to guide thinking, but not giving solutions. As the instructor observes the group, she assesses which students are learning the material and which students are still struggling. She notes innovative student-invented strategies and solutions in order to facilitate the whole group discussion.

Once the students have had the opportunity to solve the problem, individuals or groups are asked to present their strategies and solutions to the class. It is through these discussions that those students who were still struggling can gain insight that they can apply as they try to solve the problem. Since the students often approach problems differently as they are constructing their knowledge, several strategies are presented and each are discussed within the whole class. These discussions and reflections also provide the medium for the students to engage in reflective abstraction. In addition to helping the students construct their own knowledge, this approach also allows the students to expand their problem-solving abilities, to improve their ability to reason, to better their mathematical communication abilities, to create connections with prior knowledge within mathematics and in other contexts, and to flexibly move between representations of mathematical concepts.

Many people who are new to constructivism do not realize that the role of the teacher changes dramatically. In our classrooms, the focus shifts from a one-way transmission of knowledge to a discourse in which the students interact with their peers and the new information in an attempt to ease a cognitive disturbance. The teacher, no longer the sole source of new information, becomes a guide as the students work to construct their new knowledge. It is the teacher's job to structure the learning situations, to assess student progress, and modify as needed. The teacher becomes a partner in the learning process.

In order to facilitate mathematical learning, the teacher must create tasks to engage the students in their learning process. These tasks must be relevant to the students in order to maximize motivation. The tasks must also be worthwhile—there must exist a reason to solve this problem. Therefore most of the tasks are based on real-world situations. These rich mathematical tasks must allow the students an opportunity to make and test conjectures. The students need an opportunity to analyze their solution strategies as well as those of their classmates. Many of these tasks consist of an opportunity to explore the concept concretely through hands-on activities, using materials such as manipulatives. The activities lead the students from the concrete representations to the more abstract qualities of the concepts through questions requiring higher level thinking skills, mathematical reasoning, and multiple representations.

This form of teaching places a greater load on the teacher. It is admittedly so much easier to instruct students by giving them the information rather than creating tasks and asking questions to guide them to create the knowledge on their own. However, the students do not retain material as well, nor do they have a solid understanding of the underpinnings of the content, if they are simply passively accepting information. The knowledge is not theirs, but the teacher's. They can work a problem they have seen before, but they are unable to apply their knowledge to new situations. If the students have struggled to assimilate the new knowledge into their content base, they are much more likely to be able to retrieve this information and to apply it in new and different ways because they have a greater understanding of the concept and its foundations since they built it themselves.

One must note that it is very difficult to maintain a pure constructivist classroom. Therefore, we have both adopted a blended approach. This approach is predominantly guided by constructivism, but there are some more traditional aspects, such as homework and tests. However, the homework and test questions are based on conceptual understanding of the topics rather than rote manipulations of symbols to arrive at an answer. For instance, instead of having students find the slope given a linear equation, we will give them data which can be modeled by a linear function, ask them to find the function, then ask for the slope. The main difference is that we then ask them to determine the meaning of the slope in the context of the problem. The focus of the questions is on the idea of slope as a rate of change and how that affects a linear relationship rather than finding the slope of the line between two given points.

We do lecture, but rarely, and never over 15 minutes. Our students are the ones at the board showing their strategies and solutions, answering their peers' questions and justifying their solutions. Those listening to the solution then evaluate the solution and the justification, asking questions when something is unclear. We often use manipulatives or dynamic software to explore concepts and then discuss their observations. These discussions lead to the more intricate subtleties of the concept far better than a straight lecture ever could. Since the students are engaged and actively participating in the discourse, they are much more likely to construct and integrate these subtleties into their knowledge.

Through a classroom environment such as this, our students have learned that "Why?" is the most important question they can ask because it leads to greater understanding of any topic. They are focused more on the "why" than the "how." As the student answers the question of "why," they can create the process to solve the problem, which answers the question "how," because they understand the mathematical concepts underpinning the solution.

CONCLUSION

Research has indicated that successful learning involves actively rebuilding cognitive structures to accommodate new pieces of information as they are encountered. This evolutionary, dynamic form of learning is called constructivism, and it has proven to be a personally successful theory to guide teaching for both of us. While constructivism takes on many different forms, the essential core beliefs of constructivism in mathematics education can be summarized as follows:

1. Mathematical knowledge is actively constructed through a process called reflective abstraction.
2. Cognition is evolutionary: cognitive structures adapt to disturbances from novel stimuli in order to accommodate the stimuli in an ordered fashion.
3. Constructivism as a teaching practice is difficult to maintain in its purest form, but it is a beneficial style of pedagogy that puts the student, rather than the teacher, at the center of the learning process.

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