Obtaining closed form Bayes factors from summary statistics in common experimental designs

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The goal of this talk is to describe some methods for evaluating evidential value of data in analysis of variance models.

By *evidential value*, I mean the factor by which the prior odds is updated after observing data:

$$\underbrace{\frac{p(\mathcal{H}_1 \mid \mathcal{D})}{p(\mathcal{H}_0 \mid \mathcal{D})}}_{\text{posterior odds}} = \underbrace{\frac{p(\mathcal{H}_1)}{p(\mathcal{H}_0)}}_{\text{prior odds}} \times \underbrace{\frac{p(\mathcal{D} \mid \mathcal{H}_1)}{p(\mathcal{D} \mid \mathcal{H}_0)}}_{\text{predictive updating factor}}$$

Kass and Raftery (1995) called this predictive updating factor the Bayes factor

Motivating example: consider test scores from students in three instructional treatments:

Treatment 1	Treatment 2	Treatment 3
2	5	8
3	9	6
8	10	12
6	13	11
5	8	11
6	9	12
$M = 5$	$M = 9$	$M = 10$

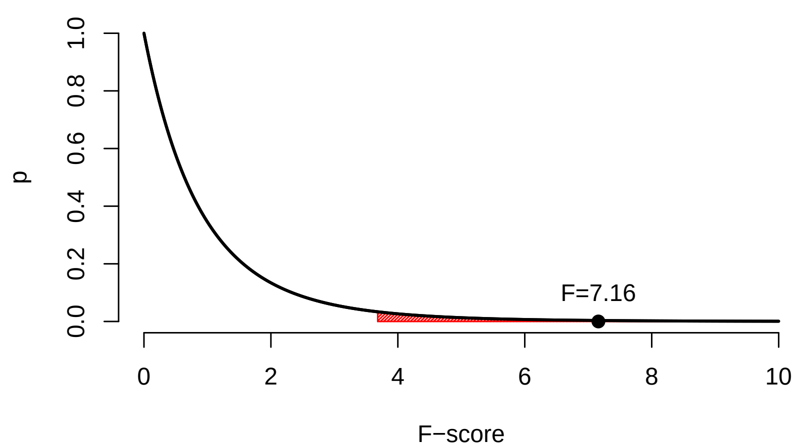
Typical question – are there differences among these condition means?

Classical approach - analysis of variance (ANOVA)

- model $Y_{ij} = \mu + \alpha_j + \varepsilon_{ij}$, where $\varepsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$
- assume "null hypothesis" $\mathcal{H}_0 : \alpha_j = 0$
- compute probability of observing data Y_{ij} under $\mathcal{H}_0$
- if data is $rare$ under $\mathcal{H}_0$, reject $\mathcal{H}_0$
- index "rareness" by computing F (a ratio of between-group and within-group variances)

ANOVA computations

source	SS	df	MS	F
between treatments	84	2	42	7.16
within treatments	88	15	5.87	
total	172	17		



Since our data Y_{ij} is rare under $\mathcal{H}_0$, we reject $\mathcal{H}_0$ as an implausible model restriction.

This ignores predictive adequacy of $\mathcal{H}_1$.

The **Bayes factor**

$$B_{10} = \frac{p(\mathcal{D} \mid \mathcal{H}_1)}{p(\mathcal{D} \mid \mathcal{H}_0)}$$

tells us how much better $\mathcal{H}_1$ predicts our observed data compared to $\mathcal{H}_0$.

Example: suppose $B_{12} = 5$. This means that the observed data are 5 times more likely under $\mathcal{H}_1$ than $\mathcal{H}_0$.

Problem – computing Bayes factors is hard!

$$\begin{aligned} B_{10} &= \frac{p(\mathcal{D} \mid \mathcal{H}_1)}{p(\mathcal{D} \mid \mathcal{H}_0)} \\ &= \frac{\int f(\mathcal{D} \mid \theta_1, \mathcal{H}_1)\pi(\theta_1 \mid \mathcal{H}_1)d\theta_1}{\int f(\mathcal{D} \mid \theta_0, \mathcal{H}_0)\pi(\theta_0 \mid \mathcal{H}_0)d\theta_0} \end{aligned}$$

Today, I'll describe two approaches to making this computation easier

1. BIC approximation (Raftery, 1995; Wagenmakers, 2007; Masson, 2011)

Basic idea – if we construct 2nd order Taylor approximation of log-marginal likelihood of each $\mathcal{H}_i$, we get

$$BF_{01} \approx \exp\left(\frac{\mathrm{BIC}(\mathcal{H}_1) - \mathrm{BIC}(\mathcal{H}_0)}{2}\right),$$

where

$$\mathrm{BIC}(\mathcal{H}_i) = n \ln\left(\frac{SSR}{SST}\right) + k \ln n.$$

In 2018[1], I derived a simple formula that computes this BIC Bayes factor using only the ANOVA summary statistics:

$$BF_{01} \approx \sqrt{n^x \left(1 + \frac{Fx}{y}\right)^{-n}},$$

where

- F is the observed F-ratio
- x, y are the numerator/denominator df, respectively
- n is the total number of observations.

[1] Faulkenberry, T.J. (2018). Computing Bayes factors to measure evidence from experiments: An extension of the BIC approximation. *Biometrical Letters, 55*, 31-43

Using the summary statistics from the ANOVA ($F = 7.16$, $x = 2$, $y = 15$, and $n = 18$), we get

$$\begin{aligned} BF_{01} &\approx \sqrt{n^x \left(1 + \frac{Fx}{y}\right)^{-n}} \\ &= \sqrt{18^2 \left(1 + \frac{7.16 \cdot 2}{15}\right)^{-18}} \\ &= 0.0432, \end{aligned}$$

Thus, $BF_{10} = 1/BF_{01} \approx 1/0.0432 = 23.15$

Bad approximation?

Sellke et al. (2001) showed that under a reasonable class of prior distributions for p-values, an upper bound for the Bayes factor can be computed directly from the p-value as

$$
\begin{aligned}
BF_{10} &\leq -\frac{1}{e \cdot p \ln(p)} \\
&=\leq -\frac{1}{e \cdot 0.0066 \cdot \ln(0.0066)} \\
&= 11.10.
\end{aligned}
$$

Thus, our BIC Bayes factor of 23.15 is quite an overestimate of the actual Bayes factor. **Can we compute an exact Bayes factor with only summary statistics?**

2. Pearson Type VI Bayes factor

In some new work[2], I derived the following *exact* Bayes factor:

$$BF_{10} = \frac{\Gamma\left(\frac{x+1}{2}\right) \cdot \Gamma\left(\frac{y}{2}\right)}{\Gamma\left(\frac{x+y}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)} \left(\frac{y}{y+xF}\right)^{\frac{1-y}{2}}.$$

[2]Faulkenberry, T. J. (2021). The Pearson Bayes factor: An analytic formula for computing evidential value from minimal summary statistics, *Biometrical Letters, 58*, 1-26. https://doi.org/10.2478/bile-2021-0001

Using our example, we have

$$
\begin{aligned}
BF_{10} &= \frac{\Gamma\left(\frac{2+1}{2}\right) \cdot \Gamma\left(\frac{15}{2}\right)}{\Gamma\left(\frac{2+15}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)} \left(\frac{15}{15 + 2 \cdot 7.16}\right)^{\frac{1-15}{2}} \\
&= \frac{0.8662269 \cdot 1871.254}{14034.41 \cdot 1.772454} (0.5116)^{-7} \\
&= 7.268
\end{aligned}
$$

Some observations:

- the resulting Bayes factor is reasonable w.r.t. the Sellke bound
- the expression is $analytic$, but not $closed\ form$

Theorem 1. *Given an ANOVA summary reported in standard form $F(x, y)$ (i.e., where x equals the between-treatments degrees of freedom and y equals the residual degrees of freedom), the Bayes factor can be expressed in closed form as*

$$BF_{10} = C\sqrt{\left(\frac{y}{y + xF}\right)^{1-y}}$$

where C depends on the parity of x and y, as follows:

Case 1: if x and y are even, then

$$C = \frac{x!\left(\frac{y}{2} - 1\right)!}{2^x\left(\frac{x}{2}\right)!\left(\frac{x+y}{2} - 1\right)!}$$

Case 2: if x is even and y is odd, then

$$C = \frac{x!(y-1)!\left(\frac{x+y-1}{2}\right)!}{\left(\frac{x}{2}\right)!\left(\frac{y-1}{2}\right)!(x+y-1)!}$$

Case 3: if x is odd and y is even, then

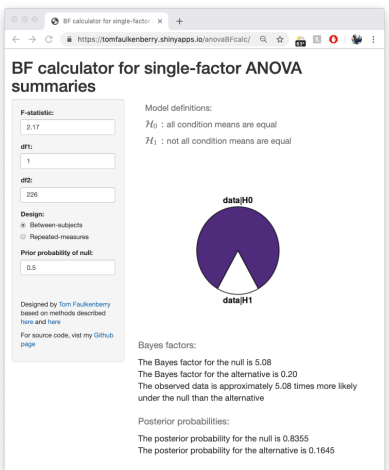
$$C = \frac{2^{x+y-1}\left(\frac{x+y-1}{2}\right)!\left(\frac{y}{2}-1\right)!\left(\frac{x-1}{2}\right)!}{\pi(x+y-1)!}$$

Case 4: if x and y are odd, then

$$C = \frac{\left(\frac{x-1}{2}\right)!(y-1)!}{2^{y-1}\left(\frac{y-1}{2}\right)!\left(\frac{x+y}{2}-1\right)!}$$

Future goals

In a recent paper[3], I described a web app that computes Bayes factors from ANOVA summaries. This has been integrated into a larger app called PsyStat – https://tomfaulkenberry.shinyapps.io/psystat



- "Shiny app" (based on R)
- gives Bayes factors and posterior probabilities
- currently uses BIC approximation
- working on integrating exact Bayes factors

[3] Faulkenberry, T. J. (2019). Estimating evidential value from analysis of variance summaries: A comment on Ly et al.(2018). *Advances in Methods and Practices in Psychological Science, 2*(4), 406-409

Thank you!

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