Modeling a latent structure of individual differences in numerical cognition

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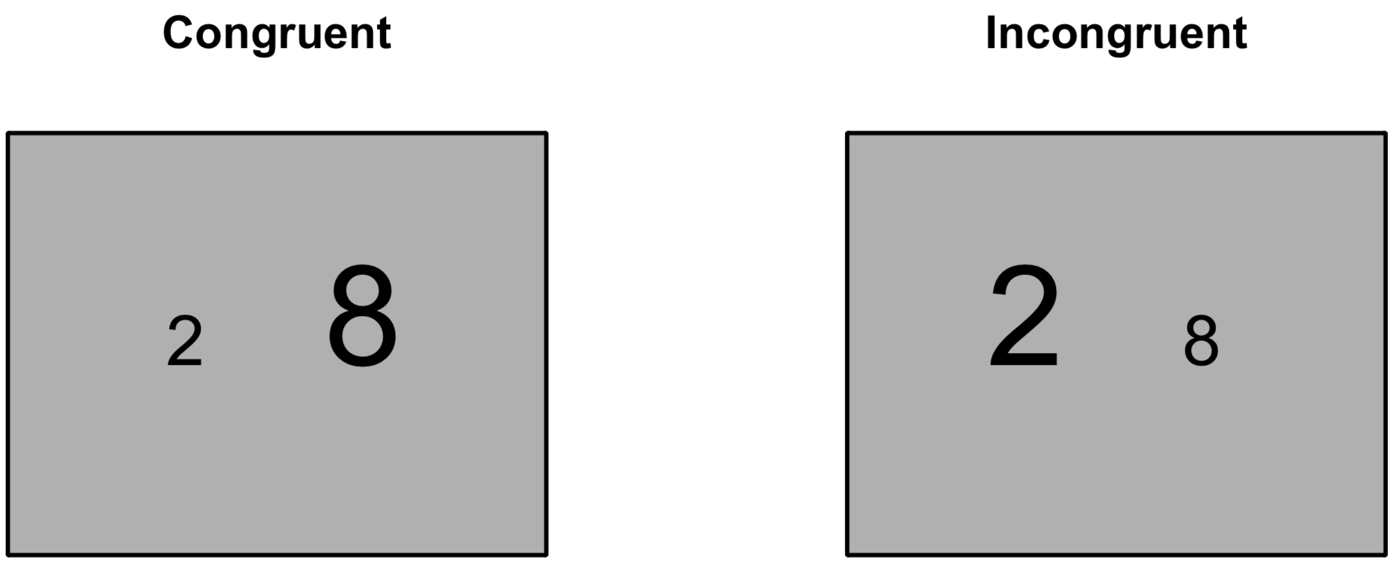
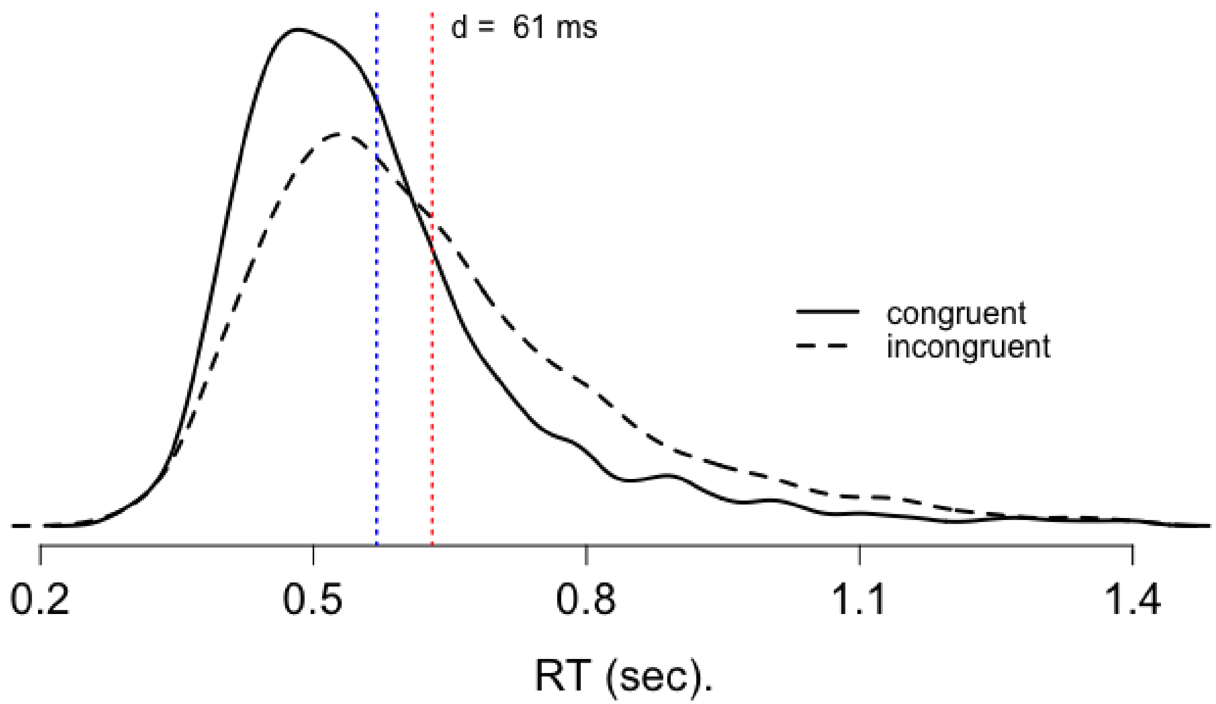
vARMADILLO 2020

Size congruity effect

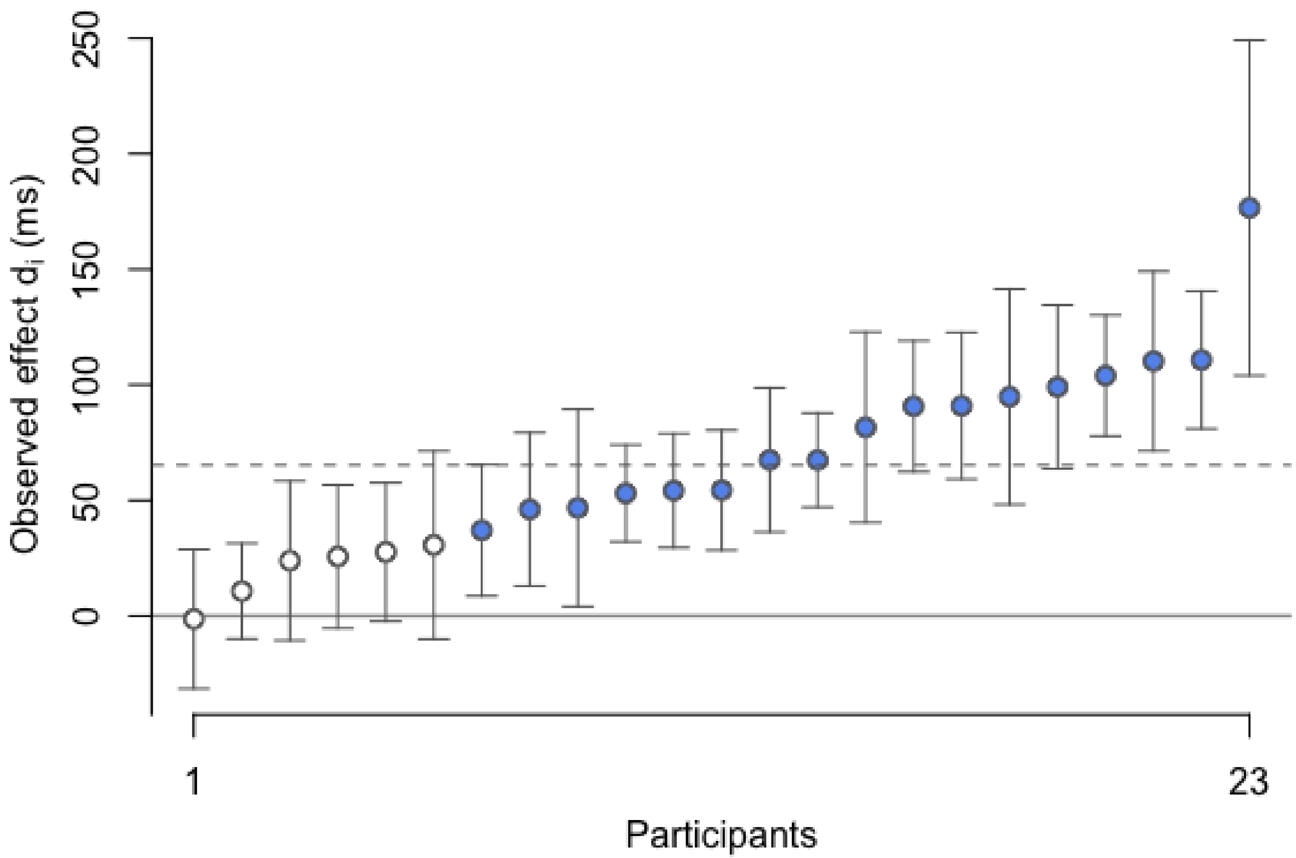
Typical laboratory task: choose the **physically larger** digit



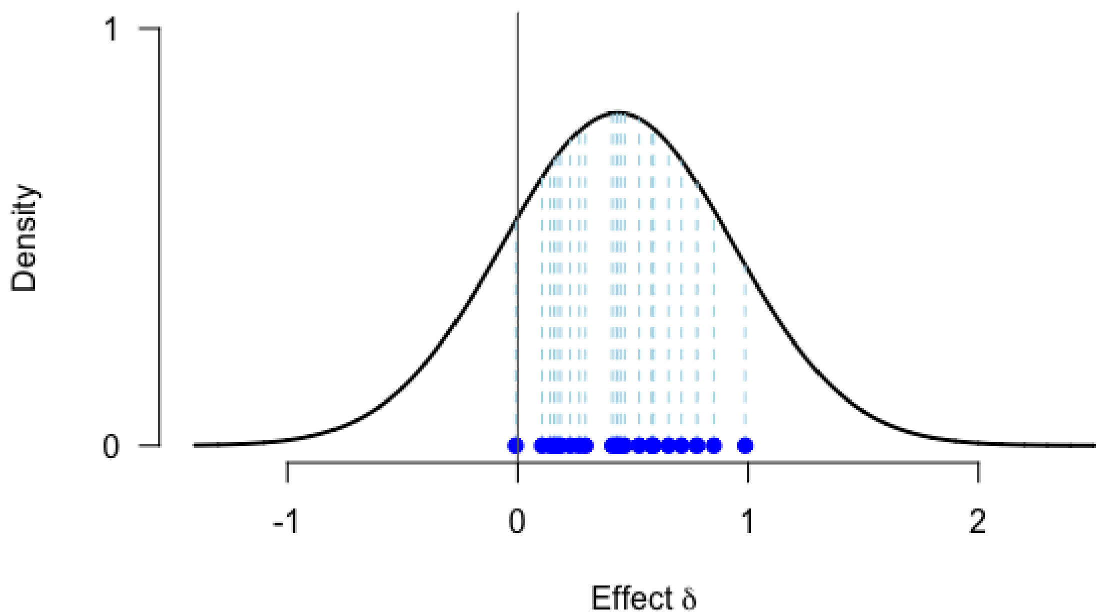
Typical result – mean RT larger for *incongruent trials*



What about individual effects?



Consider these observed effects d_i as being drawn from (population) distribution of *true* effects δ. What is the structure of this population?



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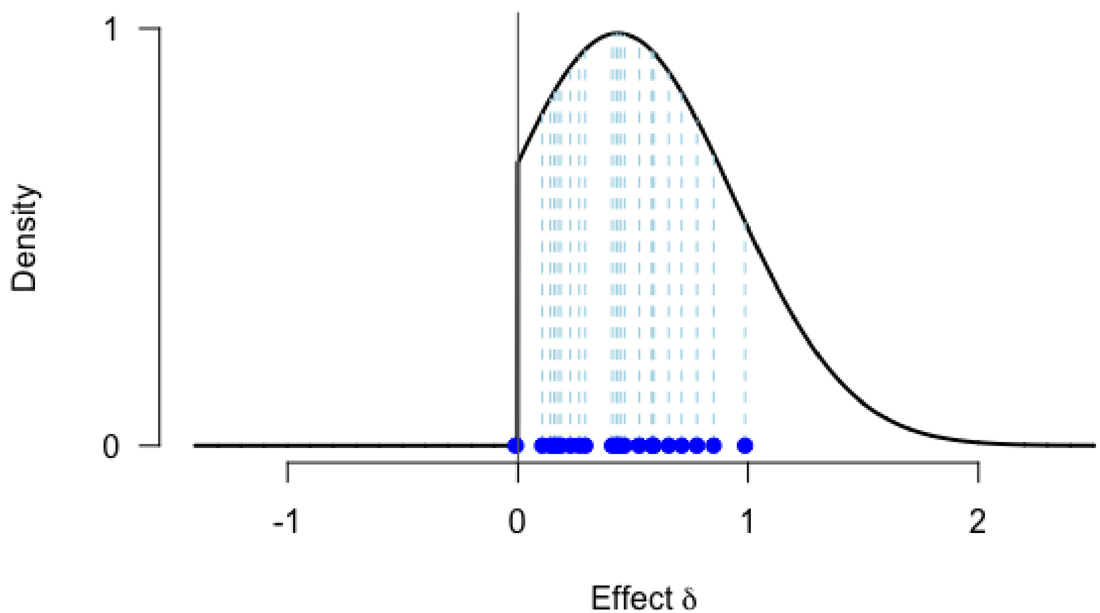


A new question

Does everybody *exhibit the size congruity effect?*

- if *yes*, then SCE is obligatory, resistant to strategic control, ...
- if *no*, then SCE is complex, malleable, ...

Importantly, both answers have downstream consequences for processing architecture of numerical cognition

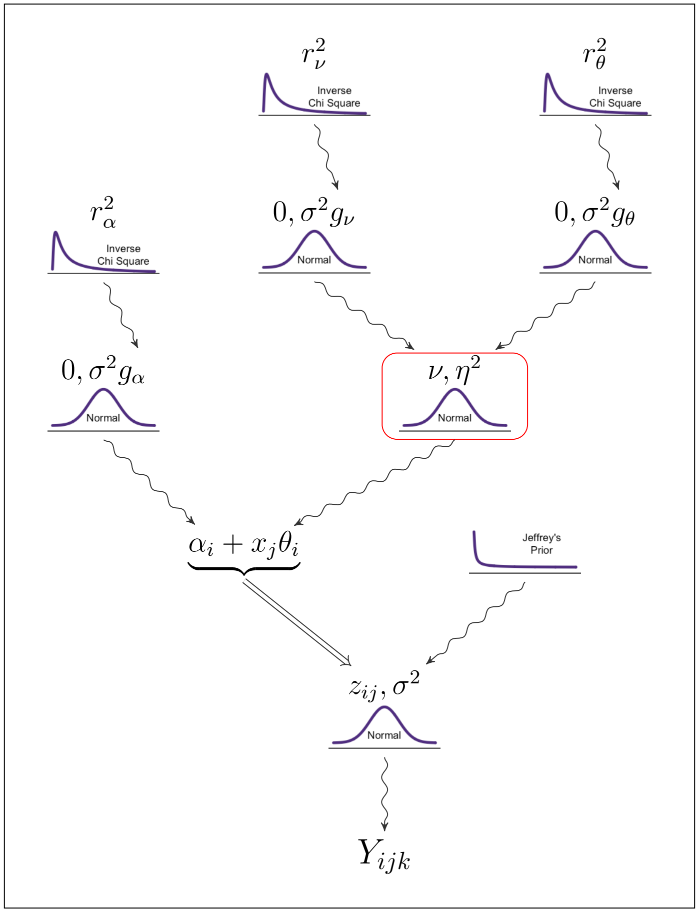
A new question

Does everybody *exhibit size congruity effect?*

How to answer:

- build competing models of individual difference structures in SCE
- adjudicate the models via Bayesian model comparison

Hierarchical structure



Basic idea (Haaf & Rouder, 2017):

- model RTs as a random-effects linear model with effect parameter θ_i $(i = 1 \ldots, N)$
- assume (as baseline) that θ_i is drawn from a normal distribution with mean ν and variance η^2
- use g-priors (Zellner & Siow, 1980), specifying *a priori* scale on variance of overall effect and individual variability around effect
- define competing models by **constraining** effect parameter θ_i

Four competing models

1. Unconstrained model, $\mathcal{M}_u$

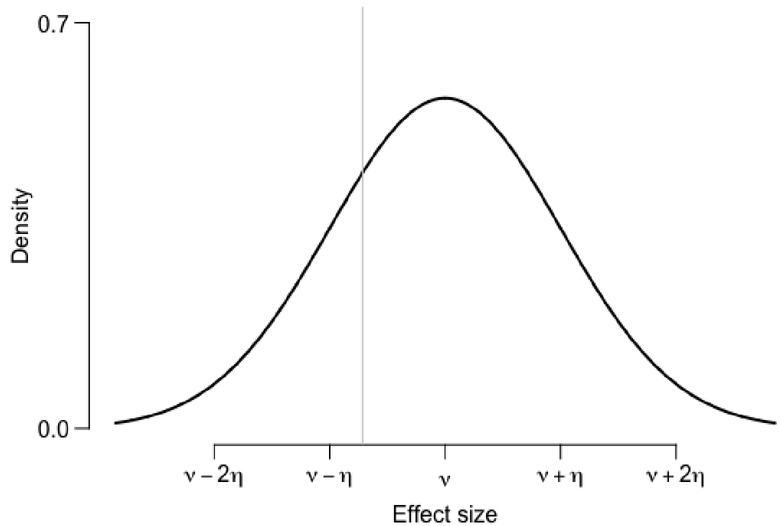
2. Positive-effects model, $\mathcal{M}_+$

3. Common-effect model, $\mathcal{M}_1$

4. Null-effect model, $\mathcal{M}_0$

Unconstrained model

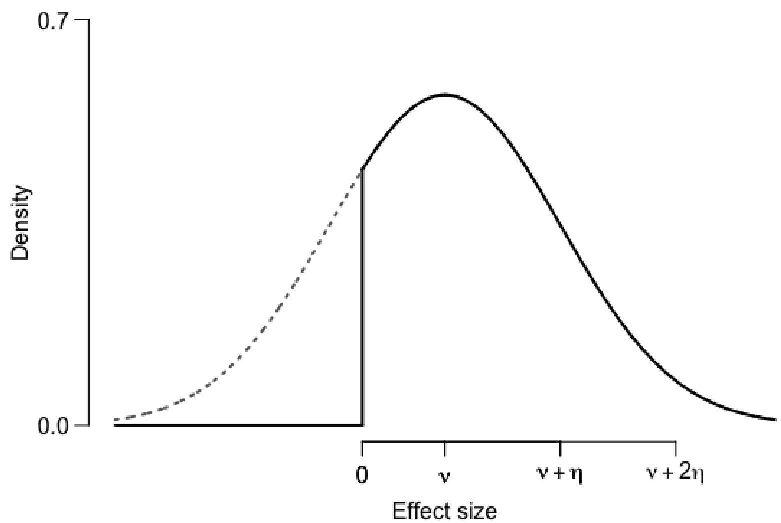
$\mathcal{M}_u : \theta_i \sim \text{Normal}(\nu, \eta^2)$



Qualitative individual differences: some positive, some negative

Positive-effects model

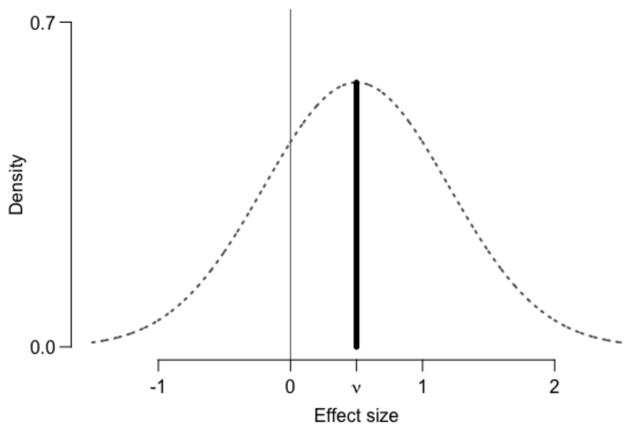
- $\mathcal{M}_+ : \theta_i \sim \text{Normal}_+(\nu, \eta^2)$



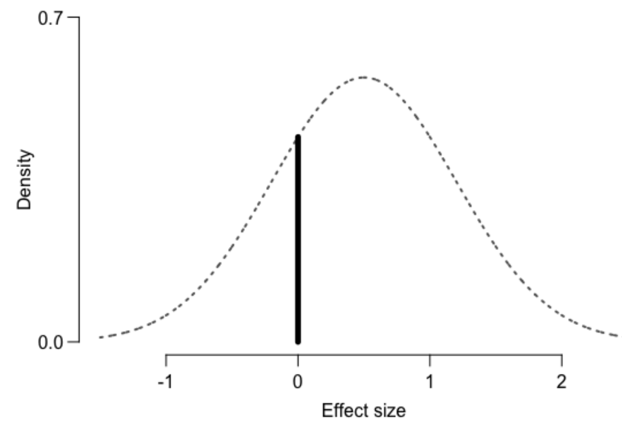
Quantitative individual differences: all positive, vary in magnitude

Common-effect / null models

- $\mathcal{M}_1 : \theta_i = \nu$



- $\mathcal{M}_0 : \theta_i = 0$



Model comparison

We use the **Bayes factor**, which indexes how well the observed data $\boldsymbol{Y}$ are predicted under one model relative to another:

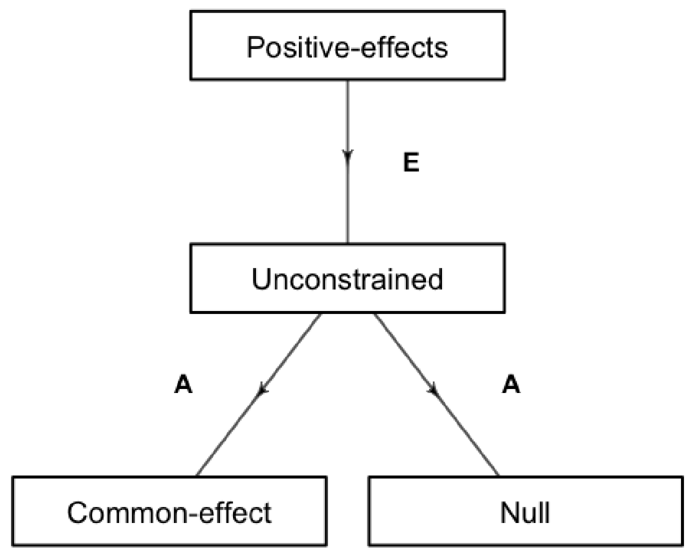
$$B_{ab} = \frac{p(\boldsymbol{Y} \mid \mathcal{M}_a)}{p(\boldsymbol{Y} \mid \mathcal{M}_b)}$$

$B_{ab} = 10$ means:

- the observed data are 10 times more likely under $\mathcal{M}_a$ compared to $\mathcal{M}_b$
- "10-to-1 evidence for $\mathcal{M}_a$ over $\mathcal{M}_b$"

Bayes factor computations

How do we compute $B_{ab} = \dfrac{p(\boldsymbol{Y} \mid \mathcal{M}_a)}{p(\boldsymbol{Y} \mid \mathcal{M}_b)}$?



Bayes factor computations

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$$p(\boldsymbol{Y} \mid \mathcal{M}) = \int_{\boldsymbol{\xi} \in \Xi} p(\boldsymbol{Y} \mid \boldsymbol{\xi}) p(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

Bayes factor computations

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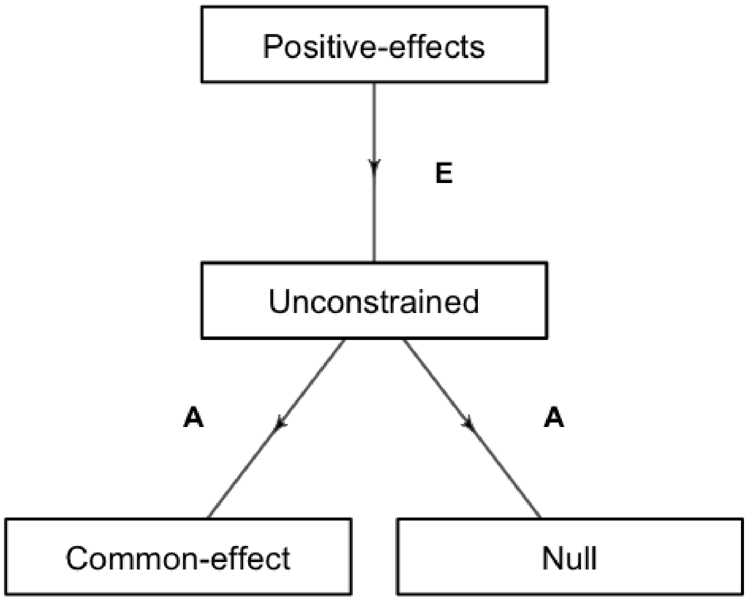
Problem: for our models $\mathcal{M}$, the parameter vectors $\boldsymbol{\xi}$ look like

$$\boldsymbol{\xi} = (\mu, \sigma^2, \nu, \alpha_1, \ldots, \alpha_N, \theta_1, \ldots, \theta_N, g_\alpha, g_\nu, g_\theta)$$

so the integral is carried out in $\mathbb{R}^{2N+6}$.

For $N = 35$, this would be a 76-dimensional integral!

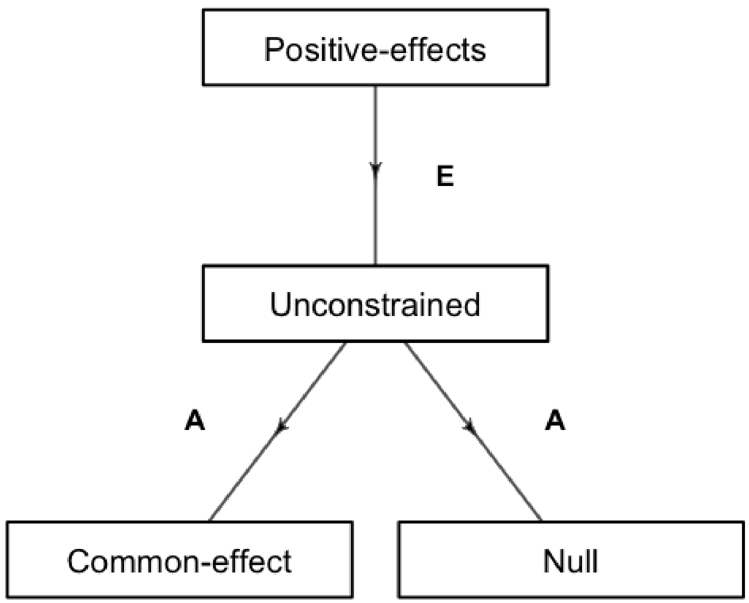
Bayes factor computations



A = analytic approach

- Zellner & Siow (1980); Rouder et al. (2012)
- place g-priors on individual intercepts and effect parameters
- everything *except* the g-parameters integrates symbolically
- g-parameters can be well approximated with MCMC sampling
- techniques coded into BayesFactor package in R

Bayes factor computations



E = encompassing approach

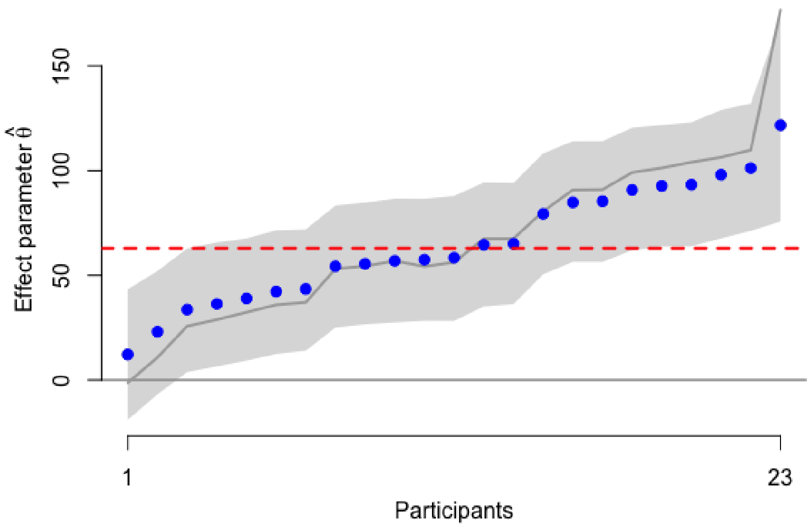
- Klugkist et al. (2005)
- $B_{+u} = \dfrac{P(\boldsymbol{\theta} > 0 \mid \boldsymbol{Y}, \mathcal{M}_u)}{P(\boldsymbol{\theta} > 0 \mid \mathcal{M}_u)}$
- probabilities computed as fraction of MCMC samples from unrestricted model that are **positive** for all individuals (both *a priori* and *a posteriori*)

Datasets

We modeled three datasets, each using the same basic size-congruity task.

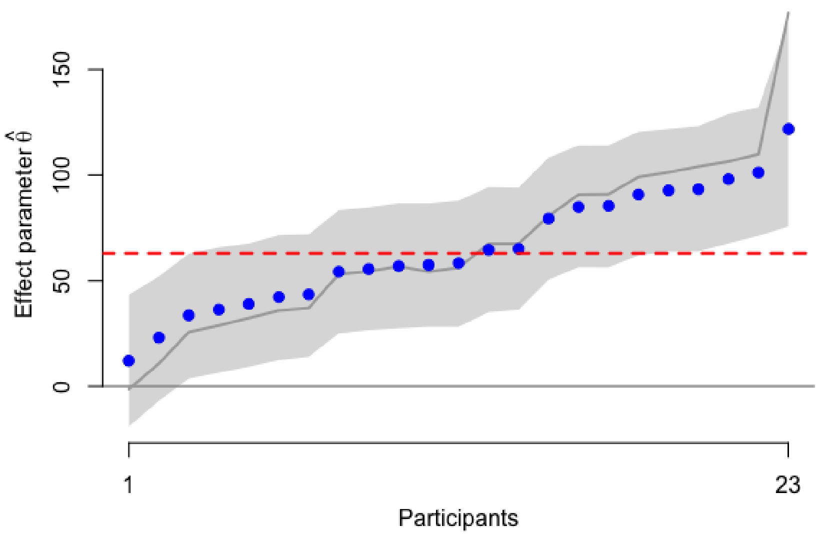
Dataset	N	# obs	Mean RT	SD	Error rate
1	23	8,832	614 ms	349 ms	5.6%
2	53	20,352	593 ms	299 ms	4.2%
3	35	6,720	674 ms	354 ms	4.5%

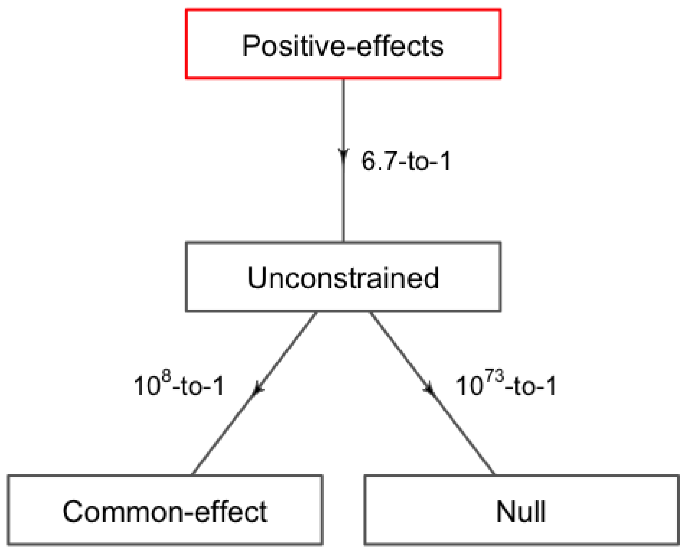
Results - Dataset 1



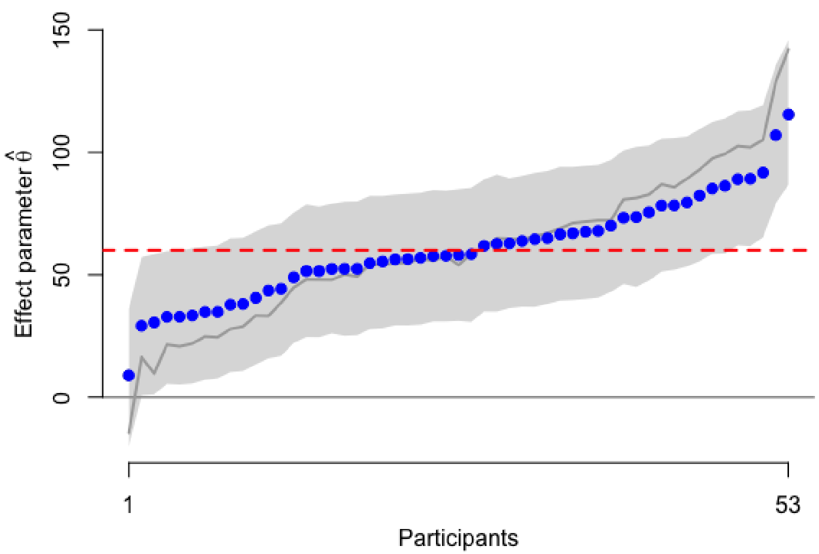
- Red line = estimated effect θ from $\mathcal{M}_1$
- Blue dots = individual effect estimates θ_i
- Gray line = estimates from mean differences d_i
- Gray area = 95% credible intervals

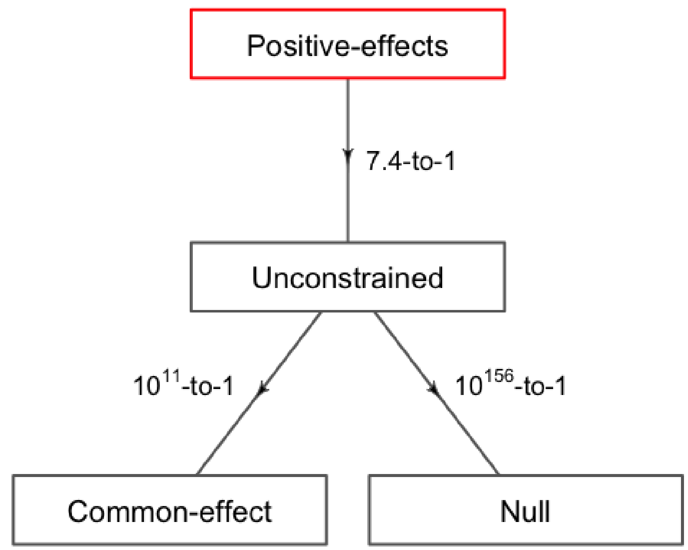
Results - Dataset 1



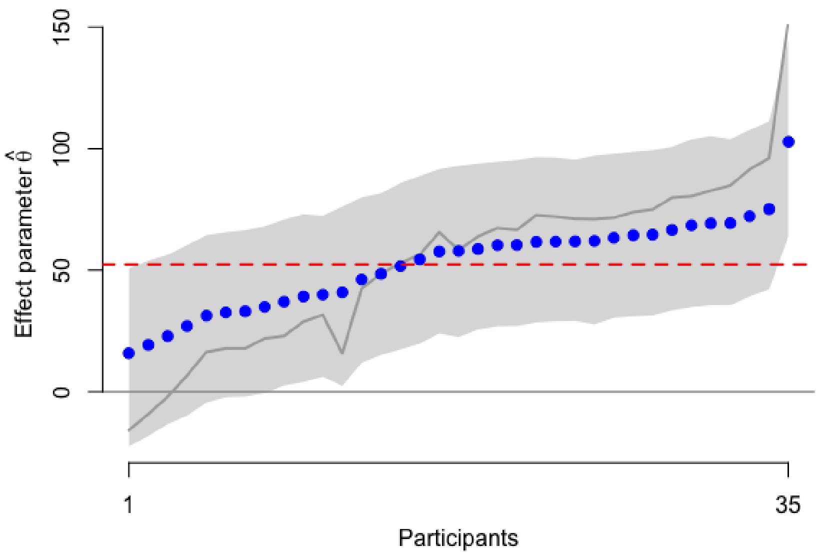


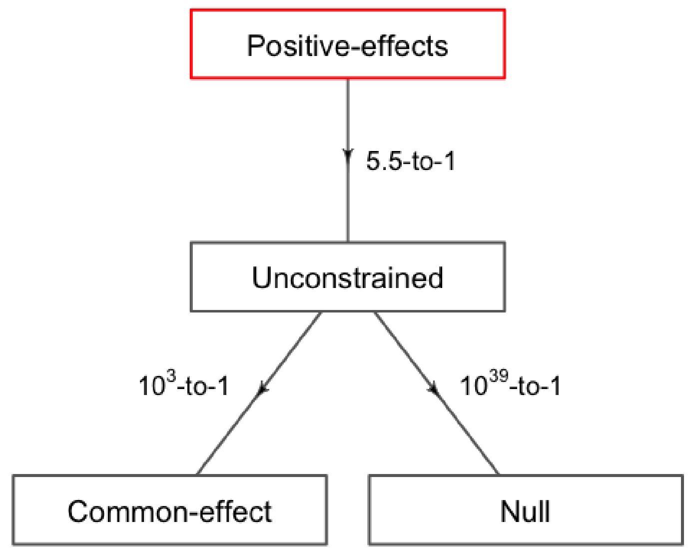
Results - Dataset 2





Results - Dataset 3





Summary points

Does everybody *exhibit the size congruity effect?*

- if *yes*, then SCE is obligatory, resistant to strategic control, . . .
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Importantly, both answers have downstream consequences for processing architecture of numerical cognition

- what does this say about *early vs. late* interaction debate (e.g., Faulkenberry et al., 2016; Sobel et al., 2016; 2017; Faulkenberry, Vick, & Bowman, 2018; Bowman, 2020)

Some other benefits:

- Bayes factors easy to interpret

- hierarchical structure removes trial noise from individual estimates

- common effect (CE) model provides important self-check:
  - if CE model is best, is our design adequate to capture individual differences

- Might be good approach to disentangle competing theories of mental arithmetic
  - Does everyone exhibit size-by-format interaction?
  - Does everyone reflect fast counting in small addition problems?

Thank you!

- slides available at `https://tomfaulkenberry.github.io/talks.html`
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